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Row–column designs for comparing treatments with a control

Dibyen Majumdar[a], Ajit C. Tamhane[b,*]

[a] *Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL 60607, USA*

[b] *Department of Statistics, Northwestern University, Evanston, IL 60208, USA*

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Abstract

This paper considers the problem of the design and analysis of experiments for comparing several treatments with a control when heterogeneity is to be eliminated in two directions. A class of row–column designs which are balanced for treatment vs. control comparisons (referred to as the balanced treatment vs. control row–column or BTCRC designs) is proposed. These designs are analogs of the so-called BTIB designs proposed by Bechhofer and Tamhane (*Technometrics* **23** (1981) 45–57) for eliminating heterogeneity in one direction. Some methods of analysis and construction of these designs are given. A measure of efficiency of BTCRC designs in terms of the A-optimality criterion is derived and illustrated by several examples.

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1. Introduction

Considerable work has been done in the last decade on block (one-way elimination of heterogeneity) designs for comparing treatments with a control beginning with the article by Bechhofer and Tamhane (1981). These authors proposed a class of designs called the balanced treatment incomplete block (BTIB) designs, which were earlier considered by Pearce (1960). An account of most of the available results is given in Hedayat et al. (1988). However, row–column (two-way elimination of heterogeneity) designs for this problem have not received as much attention. Notz (1985), Jacroux

* Corresponding author. E-mail: ajit@iems.nwu.edu. Tel.: 708 491 3974. Fax: 708 491 4939.

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(1986), Toman and Notz (1991) and Ture (1991, 1994) have considered some optimality aspects of such designs, while Rashad (1984) and Mandeli (1991) have considered some analysis and construction aspects. The primary objective of the present paper is to introduce a general class of balanced row–column designs for comparing treatments with a control (referred to as the balanced treatment vs. control row–column or BTCRC designs), and discuss some methods of analysis and construction for them. In the final section of the paper we give some methods to compute the efficiency of BTCRC designs based on the A-optimality criterion.

2. Preliminaries

Suppose that we wish to compare $v \geqslant 2$ treatments, labelled $1, 2, \ldots, v$, with a control, labelled 0, in a row–column design with $a \geqslant 2$ rows and $b \geqslant 2$ columns. Assume that only one treatment is applied in each of the $N = ab$ plots. Let y_{ijk} be the observation on the ith treatment applied in the jth row and kth column $(0 \leqslant i \leqslant v, 1 \leqslant j \leqslant a, 1 \leqslant k \leqslant b)$. We assume the usual fixed-effects additive linear model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon_{ijk}, \tag{2.1}$$

where μ is the grand mean effect, α_i is the ith treatment effect, β_j is the jth row effect, γ_k is the kth column effect and the ε_{ijk} are uncorrelated random errors with zero mean and constant variance σ^2. The various effects are assumed to satisfy the side conditions $\sum_{i=0}^{v} \alpha_i = \sum_{j=1}^{a} \beta_j = \sum_{k=1}^{b} \gamma_k = 0$.

It is of interest to estimate the treatment vs. control contrasts $\alpha_0 - \alpha_i$, $i = 1, \ldots, v$. In order for these contrasts to be estimable, a necessary condition is that $v \leqslant (a-1)(b-1)$. Let $\hat{\alpha}_0 - \hat{\alpha}_i$, $i = 1, \ldots, v$, denote the corresponding least squares estimators. As in the case of BTIB designs, we define a BTCRC design as follows.

Definition. For the above set up, a design in which a control and $v \geqslant 2$ treatments are allocated in an $a \times b$ array is a BTCRC design if the least squares estimators of the treatment vs. control contrasts satisfy

$$\mathrm{Var}(\hat{\alpha}_0 - \hat{\alpha}_i) = \tau^2 \sigma^2 \quad (1 \leqslant i \leqslant v) \tag{2.2}$$

and

$$\mathrm{corr}(\hat{\alpha}_0 - \hat{\alpha}_i, \hat{\alpha}_0 - \hat{\alpha}_{i'}) = \rho \quad (1 \leqslant i \neq i' \leqslant v), \tag{2.3}$$

where τ^2 and ρ are some constants which depend on the particular design employed.

3. Characterization of BTCRC designs

Consider a row–column design having m_{ij} incidences of the ith treatment in the jth row and n_{ik} incidences of the ith treatment in the kth column $(0 \leqslant i \leqslant v,$

$1 \leqslant j \leqslant a, 1 \leqslant k \leqslant b$). Let $M = \{m_{ij}\}$ and $N = \{n_{ik}\}$ denote the row and column incidence matrices, respectively. Further let $r_i = \sum_{j=1}^{a} m_{ij} = \sum_{k=1}^{b} n_{ik}$ be the number of replications of the ith treatment, $r = (r_0, r_1, \ldots, r_v)'$, and

$$\mu_{ii'} = \sum_{j=1}^{a} m_{ij} m_{i'j} \quad \text{and} \quad \nu_{ii'} = \sum_{k=1}^{b} n_{ik} n_{i'k}.$$

Define

$$\lambda_{ii'} = \frac{1}{ab} [a\mu_{ii'} + b\nu_{ii'} - r_i r_{i'}].$$

We then have the following theorem.

Theorem 3.1. *Necessary and sufficient conditions for a row–column design to be BTCRC (i.e., to satisfy (2.2) and (2.3)) are that*

$$\lambda_{01} = \lambda_{02} = \cdots = \lambda_{0v} = \lambda_0 \ (say), \tag{3.1}$$

$$\lambda_{12} = \lambda_{13} = \cdots = \lambda_{v-1,v} = \lambda_1 \ (say), \tag{3.2}$$

and $\lambda_0 > 0$, $\lambda_0 + v\lambda_1 > 0$. *For a BTCRC design we have*

$$\tau^2 = \frac{\lambda_0 + \lambda_1}{\lambda_0(\lambda_0 + v\lambda_1)} \tag{3.3}$$

and

$$\rho = \frac{\lambda_1}{\lambda_0 + \lambda_1}. \tag{3.4}$$

Proof. The coefficient matrix (C-matrix) for the reduced normal equations of a row–column design is given by

$$C = \text{diag}(r) - \frac{1}{b} MM' - \frac{1}{a} NN' + \frac{1}{ab} rr'. \tag{3.5}$$

Thus the entries of C are

$$c_{ii} = r_i - \frac{1}{b}\mu_{ii} - \frac{1}{a}\nu_{ii} + \frac{1}{ab} r_i^2 = r_i - \lambda_{ii} \quad (0 \leqslant i \leqslant v)$$

and

$$c_{ii'} = -\frac{1}{b}\mu_{ii'} - \frac{1}{a}\nu_{ii'} + \frac{1}{ab} r_i r_{i'} = -\lambda_{ii'} \quad (0 \leqslant i \neq i' \leqslant v).$$

The C-matrix of the treatment–control contrasts (denoted by C^*) is derived by Notz (1985). It is obtained by deleting the first row and the first column of C, and can also be written as

$$C^* = \text{diag}(r^*) - \frac{1}{b} M^* M^{*\prime} - \frac{1}{a} N^* N^{*\prime} + \frac{1}{ab} r^* r^{*\prime} \tag{3.6}$$

where a superscript * indicates that the first row is deleted from that vector or matrix. The result now follows along the lines of Theorem 3.1 of Bechhofer and Tamhane (1981); the details are omitted. □

Note that the formulas for τ^2 and ρ are analogous to those obtained for a BTIB design, but here the design parameters λ_0 and λ_1 do not have a simple interpretation that they had for BTIB designs.

4. Analysis of BTCRC designs

We next consider the analysis aspects of BTCRC designs. Let

$$T_i = \sum_{j=1}^{a} \sum_{k=1}^{b} y_{ijk} = i\text{th treatment total} \quad (0 \leqslant i \leqslant v),$$

$$A_j = \sum_{i=0}^{v} \sum_{k=1}^{b} y_{ijk} = j\text{th row total } (1 \leqslant j \leqslant a),$$

$$B_k = \sum_{i=0}^{v} \sum_{j=1}^{a} y_{ijk} = k\text{th column total } (1 \leqslant k \leqslant b).$$

Further let

$$A_i^* = \sum_{j=1}^{a} m_{ij} A_j, \qquad B_i^* = \sum_{k=1}^{b} n_{ik} B_k$$

and

$$G = \sum_{i=0}^{v} T_i = \sum_{j=1}^{a} A_j = \sum_{k=1}^{b} B_k = \text{grand total}.$$

Define the adjusted treatment total as

$$Q_i = T_i - \frac{1}{b} A_i^* - \frac{1}{a} B_i^* + \frac{1}{ab} r_i G \quad (0 \leqslant i \leqslant v).$$

Then the analysis of variance (ANOVA) for a BTCRC design is given in Table 1.

Also, it can be shown that

$$\hat{\alpha}_0 - \hat{\alpha}_i = \frac{\lambda_1 Q_0 - \lambda_0 Q_i}{\lambda_0(\lambda_0 + v\lambda_1)}. \tag{4.1}$$

The common variance $\tau^2\sigma^2$ and common correlation ρ among these contrasts can be obtained from (3.3) and (3.4), respectively. Then it is straightforward to construct Dunnett-type (1955) simultaneous confidence intervals on $\alpha_0 - \alpha_i (1 \leqslant i \leqslant v)$ using the mean square error s^2 as an estimate of σ^2 with $(a-1)(b-1) - v$ degrees of freedom; for details, see Bechhofer and Tamhane (1981).

Table 1

Source	Sum of squares	Degrees of freedom
Treatments (adjusted)	$[ab/(\lambda_0 + v\lambda_1)][(\lambda_1/\lambda_0)Q_0^2 + \sum_{i=1}^{v} Q_i^2]$	v
Rows (unadjusted)	$(1/b)\sum_{j=1}^{a} A_j^2 - G^2/ab$	$a - 1$
Columns (unadjusted)	$(1/a)\sum_{k=1}^{b} B_k^2 - G^2/ab$	$b - 1$
Error	By subtraction	$(a-1)(b-1) - v$
Total	$\sum_{i=0}^{v}\sum_{j=1}^{a}\sum_{k=1}^{b} y_{ijk}^2 - G^2/ab$	$ab - 1$

Note that these designs are not orthogonal, but are variance-balanced. This has two advantages: (i) the analysis is simple, and (ii) such designs are expected to be good based on the results available for block designs where highly efficient, and even optimal designs for a variety of criteria are usually BTIB designs; see Hedayat et al. (1988).

5. Construction of BTCRC designs

The class of BTCRC designs is very wide since theoretically at least, the parameters $r_i, \mu_{ii'}, \nu_{ii'}$ can be unequal as long as the conditions (3.1) and (3.2) are fulfilled. However, such general BTCRC designs appear extremely difficult to construct. It is easier to construct BTCRC designs that possess some additional symmetry properties, e.g., the designs that are equireplicate in treatments, i.e., $r_1 = r_2 = \cdots = r_v$. We now present several methods of constructing such restricted BTCRC designs.

Method 1: Start with a latin square of order $w > v$ and change symbols $v + 1, \ldots, w$ to the symbol 0 (control). This method was first suggested by Notz (1985). The method can be extended by starting with a Youden design (YD) or a generalized Youden design (GYD) (Kiefer 1975a) and changing some symbols to 0.

A generalization of this method is due to Jacroux (1986): Start with a BTIB design with columns as blocks and rearrange symbols in each column so that every treatment i is replicated the same number of times in each row (the common number per row may be different for different treatments), $i = 0, 1, \ldots, v$. Hedayat and Majumdar (1988) used Jacroux's technique to construct many infinite families of BTCRC designs.

Method 2: Designs obtained using Method 1 above have all $\mu_{ii'}$ equal and all $\nu_{ii'}$ equal, i.e., these designs are row as well as column balanced. However, for an equireplicate (in treatments) BTCRC design, we only need that the $(a\mu_{0i} + b\nu_{0i})$ be equal for $1 \leqslant i \leqslant v$ and the $(a\mu_{ii'} + b\nu_{ii'})$ be equal for $1 \leqslant i \neq i' \leqslant v$. If, in addition, these two quantities are equal then the resulting design is a pseudo-Youden design (PYD) introduced by Cheng (1981). Thus a BTCRC design can be constructed from a PYD in w symbols by changing symbols $v + 1, \ldots, w$ to 0.

Example 5.1. Cheng (1981) has given a 6×6 PYD for nine treatments; a similar design was first reported by Kshirsagar (1957). Replacing symbols 7, 8 and 9 by 0's we get the following BTCRC design for six test treatments in which $\mu_{i0} + \nu_{i0} = 15$ and $\mu_{ii'} + \nu_{ii'} = 5$ for $1 \leqslant i \neq i' \leqslant 6$:

$$\begin{pmatrix} 4 & 0 & 0 & 6 & 0 & 5 \\ 3 & 1 & 2 & 0 & 0 & 0 \\ 2 & 5 & 1 & 3 & 6 & 4 \\ 0 & 3 & 6 & 2 & 5 & 0 \\ 0 & 6 & 0 & 4 & 1 & 3 \\ 5 & 0 & 4 & 0 & 2 & 1 \end{pmatrix}.$$

This design has $\lambda_0 = \frac{7}{6}$ and $\lambda_1 = \frac{7}{18}$.

Method 3: The transversal of a latin square of order v is a set of v cells such that each row, column and symbol is represented exactly once in this set; see Hedayat and Seiden (1974). By changing all symbols in a transversal to 0 one obtains a BTCRC design with $a = b = v, r_1 = \cdots = r_v = v - 1$ and $r_0 = v$.

Example 5.2. Consider the following latin square of order 4 with a transversal parenthesized:

$$\begin{pmatrix} 1 & 2 & (3) & 4 \\ 3 & 4 & 1 & (2) \\ (4) & 3 & 2 & 1 \\ 2 & (1) & 4 & 3 \end{pmatrix}.$$

Then replacing the parenthesized treatments by 0 gives the following BTCRC design:

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 4 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 2 & 0 & 4 & 3 \end{pmatrix}.$$

This design has $\lambda_0 = 1$ and $\lambda_1 = \frac{11}{16}$.

Method 4: Two transversals in a latin square of order v are called parallel if they have no cell in common. Suppose that we have identified such parallel transversals. Hedayat and Seiden's (1974) method of sum composition can be applied to obtain a $(v + 1) \times (v + 1)$ BTCRC design with $r_0 = 2v + 1, r_1 = \cdots = r_v = v$ as follows. First apply Method 3 to obtain a $v \times v$ BTCRC design using the first transversal. Then take horizontal and vertical projections (Hedayat and Seiden, 1974) of the second transversal, and add a 0 to complete the design. The method will be clear from the following example.

Example 5.3. Consider the following 4×4 latin square with two parallel transversals, one parenthesized and the other square-bracketed:

$$\begin{pmatrix} [1] & 2 & (3) & 4 \\ 3 & [4] & 1 & (2) \\ (4) & 3 & [2] & 1 \\ 2 & (1) & 4 & [3] \end{pmatrix}.$$

Replace the parenthesized transversal by 0's to obtain the following BTCRC design using Method 3:

$$\begin{pmatrix} [1] & 2 & 0 & 4 \\ 3 & [4] & 1 & 0 \\ 0 & 3 & [2] & 1 \\ 2 & 0 & 4 & [3] \end{pmatrix}.$$

Next project the square-bracketed transversal horizontally and vertically, and use the sum composition method to complete the following square BTCRC design:

$$\begin{pmatrix} 0 & 2 & 0 & 4 & 1 \\ 3 & 0 & 1 & 0 & 4 \\ 0 & 3 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 3 \\ 1 & 4 & 2 & 3 & 0 \end{pmatrix}.$$

This design has $\lambda_0 = \frac{34}{25}$ and $\lambda_1 = \frac{14}{25}$.

In a similar fashion, we can obtain a $(v+t-1) \times (v+t-1)$ BTCRC design by starting from a latin square of order v with t parallel transversals.

Method 5: The patchwork method of Kiefer (1975b) can be used to construct larger BTCRC designs from smaller ones as the following example shows.

Example 5.4. Suppose that $v = 4$. Let d_1 be a 6×6 BTCRC design obtained from a latin square of order 6 by changing symbols 5 and 6 to 0,

$$d_2 = \begin{pmatrix} 1 & 0 & 3 & 4 & 2 & 0 \\ 0 & 3 & 4 & 2 & 0 & 1 \\ 4 & 2 & 0 & 0 & 1 & 3 \end{pmatrix},$$

and d_3 be a 3×3 matrix of all 0's. Then

$$\begin{pmatrix} d_1 & d_2' \\ d_2 & d_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 & 3 & 4 & 0 & 3 & 2 \\ 4 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 0 \\ 3 & 4 & 0 & 0 & 1 & 2 & 4 & 2 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 1 & 3 \\ 1 & 0 & 3 & 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 1 & 3 & 0 & 0 & 0 \end{pmatrix}$$

is a 9×9 BTCRC design for four test treatments. This design has $\lambda_0 = \frac{14}{9}$ and $\lambda_1 = \frac{14}{3}$. The BTCRC design d_2 belongs to the "Euclidean family" of Hedayat and Majumdar (1988).

Method 6: Another method that suggests itself is to take a union of columns (or rows) of two (or more) BTCRC designs to get a larger one. As is well-known, this method works for BTIB designs when we take a union of the blocks. However, this method does not, in general, work for BTCRC designs. For example, consider the following two BTCRC designs:

$$d_1 = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 3 & 0 \end{pmatrix}, \qquad d_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Then it is easily checked that their row-wise or column-wise union does not result in a BTCRC design. A condition under which this method does work is derived in Lemma 5.1.

Without loss of generality, consider two BTCRC designs, d_1 and d_2, where d_i is a $a \times b_i$ design with row and column incidence matrices given by $\boldsymbol{M}_i$ and $\boldsymbol{N}_i$, respectively, and the vector of replications given by $\boldsymbol{r}_i$ $(i = 1, 2)$. Further let $\boldsymbol{M}_i^*$, $\boldsymbol{N}_i^*$ and $\boldsymbol{r}_i^*$ be the matrices and vectors obtained from $\boldsymbol{M}_i$, $\boldsymbol{N}_i$ and $\boldsymbol{r}_i$, respectively, by deleting the row corresponding to the control (0) in each one. The C- matrix, $\boldsymbol{C}_i^*$, for treatment-control contrasts for each design is given by (3.6), and from Theorem 3.1 it follows that d_i is BTCRC if and only if $\boldsymbol{C}_i^*$ is compound symmetric.

Suppose that d is a design obtained by taking a row-wise union of d_1 and d_2. Then the row and column incidence matrices, and the replication vector of d are:

$$\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2, \qquad \boldsymbol{N} = (\boldsymbol{N}_1 \,|\, \boldsymbol{N}_2), \qquad \boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r}_2.$$

For $i = 1, 2$, let

$$\boldsymbol{H}_i = \frac{1}{b_i} \boldsymbol{M}_i^* \left(\boldsymbol{I}_a - \frac{1}{a} \mathbf{1}_a \mathbf{1}_a' \right), \tag{5.1}$$

where $\boldsymbol{I}_a$ is an identity matrix of order a and $\boldsymbol{1}_a$ is a $a \times 1$ vector of all 1's. It can be shown that the C-matrix of design d is given by

$$\boldsymbol{C}^* = \boldsymbol{C}_1^* + \boldsymbol{C}_2^* + \frac{b_1 b_2}{b_1 + b_2}(\boldsymbol{H}_1 - \boldsymbol{H}_2)(\boldsymbol{H}_1 - \boldsymbol{H}_2)'. \tag{5.2}$$

The following lemma can be proved immediately from this expression or using Theorem 2.1 of Hedayat and Majumdar (1985).

Lemma 5.1. *The difference* $\boldsymbol{C}^* - (\boldsymbol{C}_1^* + \boldsymbol{C}_2^*)$ *is nonnegative definite. Further,* $\boldsymbol{C}^* = \boldsymbol{C}_1^* + \boldsymbol{C}_2^*$ *if and only if* $\boldsymbol{H}_1 = \boldsymbol{H}_2$ *where* $\boldsymbol{H}_i$ *is defined in* (5.1). *If* $b_1 = b_2$ *then this condition becomes*

$$\boldsymbol{M}_1^* - \frac{1}{a}\boldsymbol{r}_1^*\boldsymbol{1}_a' = \boldsymbol{M}_2^* - \frac{1}{a}\boldsymbol{r}_2^*\boldsymbol{1}_a'.$$

In addition, if $\boldsymbol{r}_1^* = \boldsymbol{r}_2^*$ *then this condition becomes* $\boldsymbol{M}_1^* = \boldsymbol{M}_2^*$.

Example 5.5. As an illustration, if d is an $a \times a$ BTCRC design obtained from a latin square of order a then the $a \times na$ array obtained by taking a row wise union of n copies of d is also a BTCRC design.

Example 5.6. As a second example, consider three BTCRC designs:

$$d_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad d_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{pmatrix}.$$

All three designs are equivalent for estimating treatment–control contrasts because they have the same $\boldsymbol{C}^*$ matrix. Let d be a row-wise union of d_1 with itself, let d' be a row-wise union of d_1 with d_2 and let d'' be a row-wise union of d_1 with d_3. Then it is easy to check that d and d' are BTCRC with $(\lambda_0, \lambda_1) = (\frac{4}{3}, \frac{2}{3})$ and $(\frac{4}{3}, \frac{11}{18})$, respectively, but d'' is not a BTCRC design. Using (5.2) we can calculate the C-matrices of treatment vs. control contrasts for these three designs which are as follows:

$$\boldsymbol{C}_d^* = 2\boldsymbol{C}^* = \frac{4}{9}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix},$$

$$\boldsymbol{C}_{d'}^* = \frac{1}{9}\begin{pmatrix} 23 & -\frac{11}{2} & -\frac{11}{2} \\ -\frac{11}{2} & 23 & -\frac{11}{2} \\ -\frac{11}{2} & -\frac{11}{2} & 23 \end{pmatrix},$$

$$\boldsymbol{C}_{d''}^* = \frac{1}{54}\begin{pmatrix} 120 & -24 & -24 \\ -24 & 121 & -25 \\ -24 & -25 & 121 \end{pmatrix}.$$

The C-matrices of d and d' are compound symmetric while that of d'' is not (although departure from compound symmetry is only slight). Note that both $C^*_{d'} - C^*_d$ and $C^*_{d''} - C^*_d$ are nonnegative definite, and hence d' and d'' are at least as good as d based on the usual design optimality criteria. For example, using the A-optimality criterion, we see that $\mathrm{tr}(C^{*-1}_d) = 1.50$, $\mathrm{tr}(C^{*-1}_{d'}) = 1.38$ and $\mathrm{tr}(C^{*-1}_{d''}) = 1.49$. Thus this example suggests that when taking a row-wise union of two identical latin squares, it is better to permute the rows of the latin square.

Example 5.7. Consider three BTCRC designs:

$$d_1 = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{pmatrix}, \quad d_3 = \begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 1 \end{pmatrix}.$$

Let d denote the row-wise union of d_1 and d_2, and let d' denote the row-wise union of d_1 and d_3. Both d and d' are BTCRC even though they do not satisfy the conditions of Lemma 5.1 (since $M^*_1 - (1/a)r^*_1 1'_a \neq M^*_2 - (1/a)r^*_2 1'_a$ and $M^*_1 \neq M^*_3$). It is clear that there are many ways of combining BTCRC designs to obtain a new BTCRC design.

Method 7: Stewart and Bradley (1991) gave several methods for constructing universally optimal $a \times b$ row-column designs in v treatments, but with the restriction-that some cells in the $a \times b$ array (called "empty nodes") are not involved in the experiment. They gave three design classes. If one begins with any of Stewart and Bradley's designs, and fills up the empty nodes with 0's then the result is a BTCRC design.

Example 5.8. The following BTCRC design with $v = 8$, $a = 14$, $b = 8$ is obtained from Table 6 of Stewart and Bradley (1991) using the above method.

$$\begin{pmatrix} 0 & 0 & 0 & 5 & 0 & 3 & 2 & 1 \\ 3 & 0 & 0 & 0 & 6 & 0 & 4 & 2 \\ 5 & 4 & 0 & 0 & 0 & 7 & 0 & 3 \\ 0 & 6 & 5 & 0 & 0 & 0 & 1 & 4 \\ 2 & 0 & 7 & 6 & 0 & 0 & 0 & 5 \\ 0 & 3 & 0 & 1 & 7 & 0 & 0 & 6 \\ 0 & 0 & 4 & 0 & 2 & 1 & 0 & 7 \\ 8 & 0 & 0 & 2 & 0 & 5 & 3 & 0 \\ 4 & 8 & 0 & 0 & 3 & 0 & 6 & 0 \\ 7 & 5 & 8 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 6 & 8 & 0 & 0 & 5 & 0 \\ 6 & 0 & 2 & 7 & 8 & 0 & 0 & 0 \\ 0 & 7 & 0 & 3 & 1 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 2 & 8 & 0 \end{pmatrix}.$$

This design has $\lambda_0 = \frac{98}{23}$ and $\lambda_1 = \frac{33}{112}$.

6. Efficiency of BTCRC designs

Given the dimensions of the array, $a \times b$, and the number of treatments, v, there are often several possible BTCRC designs. How to choose among these? One possible approach is to choose a design that minimizes the $\mathrm{Var}(\hat{\alpha}_0 - \hat{\alpha}_i) \propto \tau^2$, i.e., use the A-optimality criterion. In this section we give a lower bound on τ^2 which is a function of a, b and v only using the technique of the refinement of the model (Magda, 1980; Kunert, 1983). We illustrate the use of this bound to compute the efficiency of BTCRC designs. It should be noted that the bound applies to all row–column designs, not just to BTCRC designs; hence the efficiency is in the class of all row–column designs with given a, b and v.

Let $D(v, a, b)$ denote the set of all row–column designs of size $a \times b$ for v treatments and a control. Let $\mathrm{BD}(v, a, b)$ denote the class of all block designs consisting of v treatments and a control in b blocks of size a each. For a design $d \in D(v, a, b)$, consider a block design $d_1 \in \mathrm{BD}(v, a, b)$ formed by *columns* of d as blocks and another block design $d_2 \in \mathrm{BD}(v, b, a)$ formed by *rows* of d as blocks.Let $\boldsymbol{r}_d$, $\boldsymbol{N}_d$, $\boldsymbol{M}_d$ and $\boldsymbol{C}_d$ be the quantities associated with design d. Then the C-matrices of d_1 and d_2 are given by

$$\boldsymbol{C}_1 = \mathrm{diag}(\boldsymbol{r}_d) - \frac{1}{a} \boldsymbol{N}_d \boldsymbol{N}'_d$$

and

$$\boldsymbol{C}_2 = \mathrm{diag}(\boldsymbol{r}_d) - \frac{1}{b} \boldsymbol{M}_d \boldsymbol{M}'_d,$$

respectively. It follows from (3.5) that $\boldsymbol{C}_1 - \boldsymbol{C}_d$ and $\boldsymbol{C}_2 - \boldsymbol{C}_d$ are nonnegative definite. Hence it follows that

$$\mathrm{tr}(\boldsymbol{C}_d^{*-1}) \geqslant \mathrm{tr}(\boldsymbol{C}_1^{*-1}) \quad \text{and} \quad \mathrm{tr}(\boldsymbol{C}_d^{*-1}) \geqslant \mathrm{tr}(\boldsymbol{C}_2^{*-1}), \tag{6.1}$$

where each $\boldsymbol{C}^*$ matrix is obtained from the respective $\boldsymbol{C}$ matrix by deleting the row and column corresponding to the control. A lower bound on $\mathrm{tr}(\boldsymbol{C}_1^{*-1})$ can be obtained from a result of Jacroux and Majumdar (1989) (which is a generalization of a result of Majumdar and Notz (1983)) as follows: Let $L = \{1, 2, \ldots, \mathrm{int}[a/2]b\}$ where $\mathrm{int}[\cdot]$ denotes the largest integer function. For $l \in L$, let

$$e(l) = \mathrm{int}[(ab - l)/bv],$$

$$f(l) = v(a-1) + a - 2ve(l),$$

$$g(l) = vab(a-1) + vbe(l)(v - 2a + ve(l)),$$

$$h(l) = b(\mathrm{int}[l/b])^2 + 2(\mathrm{int}[l/b])(l - \mathrm{int}[l/b]b) + (l - \mathrm{int}[l/b]b),$$

$$F(l) = av\{al - h(l)\}^{-1} + av(v-1)^2\{g(l) - lf(l) + h(l)\}^{-1}.$$

Let

$$F_{\min}(v, a, b) = \min_{l \in L} F(l).$$

Then for any design $d_1 \in \mathrm{BD}(v, a, b)$,

$$\mathrm{tr}(\boldsymbol{C}_1^{*^{-1}}) \geqslant F_{\min}(v, a, b).$$

Likewise by exchanging a and b we get for any design $d_2 \in \mathrm{BD}(v, b, a)$,

$$\mathrm{tr}(\boldsymbol{C}_2^{*^{-1}}) \geqslant F_{\min}(v, b, a).$$

Hence from (6.1) it follows that for any design $d \in D(v, a, b)$,

$$\mathrm{tr}(\boldsymbol{C}_d^{*-1}) \geqslant \max\{F_{\min}(v, a, b), F_{\min}(v, b, a)\}. \quad (6.2)$$

If design $d \in D(v, a, b)$ is a BTCRC design then $\mathrm{tr}(\boldsymbol{C}_d^{*-1}) = v\tau_d^2$. Therefore

$$\tau_d^2 \geqslant (1/v)\max\{F_{\min}(v, a, b), F_{\min}(v, b, a)\} \equiv F^*(v, a, b).$$

Hence we define the efficiency of d by

$$E_d = \frac{F^*(v, a, b)}{\tau_d^2}.$$

Note that this is a conservative measure of efficiency since the actual minimum value of $\mathrm{tr}(\boldsymbol{C}_d^{*^{-1}})$ among all row-column designs $d \in D(v, a, b)$ is greater than or equal to $vF^*(v, a, b)$.

Another bound on $\mathrm{tr}(\boldsymbol{C}_d^{*^{-1}})$ is available from Notz (1985, Lemma 2.6). We calculated this bound for many examples including the ones given below, and found it to be never sharper than bound (6.2). The two bounds coincide for the cases covered by Corollary 2.1 of Notz (1985) where the optimal designs are orthogonal with respect to both rows and columns.

Example 6.1. Let $v = 4$, $a = b = 5$. Then $F^*(4, 5, 5) = 0.375$. If d_1 is the BTCRC design of Example 5.3 then $\tau_{d_1}^2 = 0.392$. Thus the efficiency of d_1 is $E_{d_1} = 95.7\%$.

Let d_2 be a 5×5 latin square in five treatments, $0, 1, \ldots, 4$. Then $\tau_{d_2}^2 = 0.4$ giving an efficiency $E_{d_2} = 93.8\%$. Thus d_2 is less efficient than d_1.

Example 6.2. Let $v = 3$, $a = 3$, $b = 6$. Then $F^*(3, 3, 6) = 0.45$. Consider the two BTCRC designs, d and d', of Example 5.6. We can readily calculate $\tau_d^2 = 0.5$ and $\tau_{d'}^2 = 0.4605$, the corresponding efficiencies being $E_d = 90\%$ and $E_{d'} = 97.7\%$. Clearly, d' is more efficient than d as noted earlier. This analysis gives us the additional information that d' is highly efficient in the class of all row–column designs $D(3,3,6)$.

Example 6.3. Let $v = 6$, $a = b = 6$. Then $F^*(6, 6, 6) = 0.346$. The BTCRC design d obtained from a PYD in Example 5.1 has $\tau_d^2 = 0.381$, and hence its efficiency is $E_d = 90.8\%$.

Example 6.4. Let $v = 8$, $a = 14$, $b = 8$. Then $F^*(8, 14, 8) = 0.134$. The BTCRC design d given in Example 5.8 has $\tau_d^2 = 0.158$ and hence its efficiency is $E_d = 84.8\%$.

A bound that is generally more conservative but easier to compute than (6.2) can be constructed by considering a design without row and column effects (Kurotschka, 1978)). Alternatively use the fact that $\mathrm{diag}(\boldsymbol{r}_d) - (1/ab)\boldsymbol{r}_d\boldsymbol{r}_d' - \boldsymbol{C}_d$ is nonnegative definite for any design $d \in D(v, a, b)$. This gives

$$\mathrm{tr}(\boldsymbol{C}_d^{*^{-1}}) \geqslant \frac{(v + \sqrt{v})^2}{ab}.$$

Hence an alternative measure of efficiency of a BTCRC design d is

$$E'_d = \frac{(v + \sqrt{v})^2}{abv\tau_d^2} = \frac{(\sqrt{v} + 1)^2}{ab\tau_d^2}.$$

Note that $E_d \geqslant E'_d$ and hence E'_d is a more conservative measure of efficiency.

Example 6.5. Consider a square ($a = b = s > v$) BTCRC design d obtained from a latin square of order s by changing symbols $v + 1, \ldots, s$ to 0 (control). For this design $r_1 = \cdots = r_v = s$ and $r_0 = s(s - v)$; also $\tau_d^2 = (s - v + 1)/\{s(s - v)\}$. Hence

$$E'_d = \frac{(\sqrt{v} + 1)^2(s - v)}{s(s - v + 1)}.$$

For $s = 5$ and $v = 4$ we get the design d_2 of Example 6.1 for which $E'_{d_2} = 90\%$. As another example, consider a 10×10 BTCRC design for $v = 8$ test treatments obtained from a latin square of order 10. For this design $E'_d = 97.7\%$. Since $E_d \geqslant E'_d$ this design is highly efficient.

Finally consider a 6×6 BTCRC design obtained from a latin square of order 6 by changing symbols 5 and 6 to 0. Then $E'_d = 100\%$ showing that this design is A-optimal in the class of all row-column designs $D(4, 6, 6)$. The A-optimality of this design also follows from the results of Notz (1985). Additional results and examples of A-optimal BTCRC designs are available in Hedayat et al. (1988) and Ting and Notz (1987).

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